Self-organized random walks and stochastic sandpile: From linear to branched avalanches

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In a model of self-organized criticality unstable sites discharge to just one of their neighbors. For constant discharge ratio α and for a certain range of values of the input energy, avalanches are simple branchless Pólya random walks, and their scaling properties can be derived exactly. If α fluctuates widely enough, avalanches become branched, due to multiple discharges, and behave like those of the stochastic sandpile. At the threshold for branched behaviour, peculiar scaling and anomalous diffusive transport are observed.

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An avalanche is a cascade of a large number of microscopic events. Generally it is triggered by a single event at a point. This creates similar events in the neighbourhood which activate the neighbours at further distances. Thus, an avalanche is a simple example of branched growth process, of which several physical examples exist [1]. Particularly interesting are situations in which the avalanche sizes have a scale free distribution, typically a simple power law, signifying the existence of long-range correlations as in critical phenomena. This is the case of avalanches in the phenomenon of Self-Organized Criticality (SOC) [2] where long-ranged spatio-temporal correlations spontaneously emerge in non-equilibrium steady states of slowly driven systems with nonlinear local relaxation mechanisms [3,4].

Sandpiles are prototype models of SOC. n_i grains reside at the i-th site of a regular lattice. The role of the external drive is to trigger transport processes through the system by adding single grains of sand at a time at randomly selected sites: $n_i \to n_i + 1$. If $n_i \geq n^c$ the sand column at i topples and grains are distributed to the neighbourhood. In the BTW model, the grain distribution process is deterministic since each neighbouring site gets one grain [2]. In the stochastic two-state sandpile model grains are transferred to randomly chosen neighbouring sites [5]. In spite of their similarities, the BTW model and the stochastic sandpile are now believed to belong to different universality classes [6]. Indeed, very recent studies show that the BTW model has a multifractal behaviour [7,8], whereas standard finite size scaling works well for the two-state stochastic sandpile [8–11].

Characterizing features of all these sandpile models are (i) the normal diffusive dynamics of the particles and (ii) the branching of the toppling process. A grain moves a unit distance in a toppling and its resultant motion under different topplings in different avalanches is diffusive. This implies that the average number of topplings in an avalanche grows as a quadratic power of the system size, L, in all isotropic sandpile models. Secondly in all models, the toppling condition is made in such a way that a single toppling can excite more than one neighbors, which ensures that an avalanche is a branched process. Normal diffusive transport and branching are believed to be very basic ingredients of SOC.

In this paper, by studying an original energy activation model with stochastic discharge mechanism, we find that neither of these ingredients (i) and (ii) as stated above, is necessary for SOC behaviour. Indeed, we show that avalanches in SOC can be linear as branchless random walks. In this case their scaling behaviour is different from that of branched avalanches, Eulerian random walk models are also linear avalanche models studied before [12]. and the relative probability distributions do not become independent of L, as this tends to infinity. By allowing the discharge ratio to fluctuate in a progressively wider interval, we are also able to trigger a branched avalanche behaviour, which falls in the universality class of the stochastic sandpile [5]. Right at threshold for this branched behaviour the diffusive transport becomes anomalous, i.e. the average avalanche size grows as a power higher than 2 of L.

Our model has two parameters: the input amount δ and the discharge ratio α. An amount of energy ϵ_i, which can vary continuously, is accumulated at each site i of a square lattice box of size L. The system is driven by injecting an amount of energy δ to a randomly selected site. Every lattice site has a limiting capacity $\epsilon_c (= 1)$ for the maximum amount of energy storage. If at any site the energy $\epsilon_i > \epsilon_c$, the site i activates and undergoes a relaxation process. In a relaxation a fraction α of the site energy is discharged to only *one* of the neighbouring sites, j, which is selected randomly: $\epsilon_j \to \epsilon_j + \alpha\epsilon_i$. The rest of the energy remains in the relaxing site: $\epsilon_i \to (1-\alpha)\epsilon_i$. If the energy at the receiving site j exceeds the threshold, that site also relaxes. The possibility of multiple relaxations arises when site i remains active even after relaxing. In such a case further relaxations occur in the subsequent stages. Upon relaxing, an active site at the boundary may drop a fraction α of its energy outside the system. This ensures that the system reaches unique

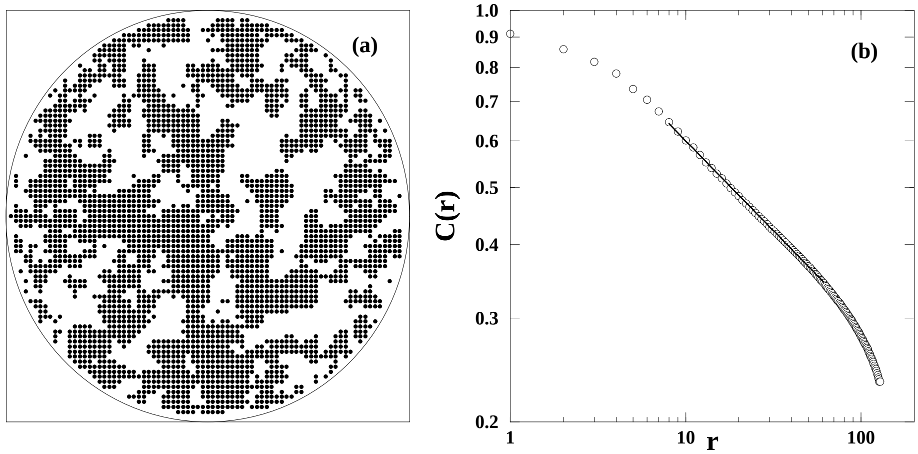

FIG. 1. (a) A steady state energy configuration in the LAM ($\alpha = 1/3$, $\delta = 1/5$) within a circular region on a square lattice of size $L = 81$. (b) Plot of the correlation function $C(r)$ for the same LAM as in (a) but for $L = 257$.

stationarity after repeated driving. Our simulations show that this is the case, independent of starting configuration as required in SOC, for any particular selection of (α,δ). We note that this model is a special case of the class of models called the abelian distributed processors model [13].

First we consider a constant value of α. In the steady state, $1-\alpha \le \epsilon_i \le 1$ since each site relaxes at least once. This implies that a site must activate after receiving at least energy α due to an activation at the neighbouring site. Energy activation events then take place one after another like in a step sequence of a Pólya random walk (RW) [14]. Once started, such a RW must terminate at the boundary, after dropping some energy outside the system. We call this the linear avalanche model (LAM).

When $\delta \le \alpha$ there must be some sites which do not activate upon receiving the input energy, since a RW drops at least α energy outside the system. To eliminate corner effects we use a circular region, within a square box of size L (odd), of radius $R = (L-1)/2$. Fig. 1(a) shows an energy configuration in the steady state where the sites having energy larger than the average, $\langle\epsilon\rangle$, are represented by black dots. Correlated regions of such sites are observed as connected clusters of black dots. The correlation function $C(r)$ is defined as the probability that a black dot at a distance r from the centre belongs to the same cluster containing a black dot at the centre. We observe a power law decay $C(r) \sim r^{-\xi}$ with $\xi = 0.37 \pm 0.02$ which is the signature of the critical correlation developed in the steady state (Fig. 1(b)). However a site energy correlation function like: $\langle e(0)e(r)\rangle - \langle e^2\rangle$ averaged over all sites has an almost uniform small negative value except when $r \sim R$.

The probability density of the site energies $D(\epsilon, R)$ in the steady state is very well fitted with a generalized Lorentzian peak in the range $1-\alpha < \epsilon < 1$ around its average $\langle\epsilon\langle$. D follows a scaling with R of the form:

$$D(\epsilon)/R = a^2/\{(\epsilon - \langle\epsilon\rangle)^2 R^2 + b^2\}^c \tag{1}$$

where, for example, $a \approx 0.195, b \approx 0.226$ and $c \approx 1.33$, with $\alpha = 1/3$ and $\delta = 1/5$ (Fig. 2). From Fig. 2(a) we see that the site energy distribution is symmetric about its average value at $\langle\epsilon\rangle$ and the most probable value coincides with the average value. Assuming that the $\epsilon >$ is less than $1-\delta$, the fraction of sites which donot activate on receiving the input energies are greater than those which activate. This situation cannot be stable since these low energy ($\epsilon < 1-\delta$) sites absorb energy at a greater rate from the external drive than the high energy sites ($\epsilon > 1-\delta$) and the average energy will push up. Similarly if $\langle\epsilon\rangle$ is greater than $1-\delta$ there would be faster dissipation of energy through the boundary which will push down the average energy. Therefore it is likely that the stable states correspond to $\langle\epsilon\rangle = 1-\delta$ where the rates of absorption by the low energy sites and the rate of dissipation by the inputs at the high energy sites are equal. Our numerical results strongly supports this result:

$$\langle\epsilon(\alpha,\delta)\rangle = 1-\delta \quad \text{for} \quad \delta \le \alpha \tag{2}$$

For $\alpha \le \delta \le \delta_c(\alpha)$ every RW drops exactly an amount of energy δ outside the boundary. $D(\epsilon, R)$ is a delta function at its average value $\langle\epsilon\rangle$ such that a site after activation retains the same energy which it had before receiving the external input δ. This implies

$$(1-\alpha)(\langle\epsilon\rangle + \delta) = \langle\epsilon\rangle \tag{3}$$

which gives $\langle\epsilon(\alpha,\delta)\rangle = (1/\alpha - 1)\delta$. The average energy increases to 1 when the input energy is increased to $\delta_c(\alpha) = \alpha/(1-\alpha)$. This is the limiting situation for the branchless avalanches. Beyond this limit multiple discharges start and avalanches cease to be branchless.

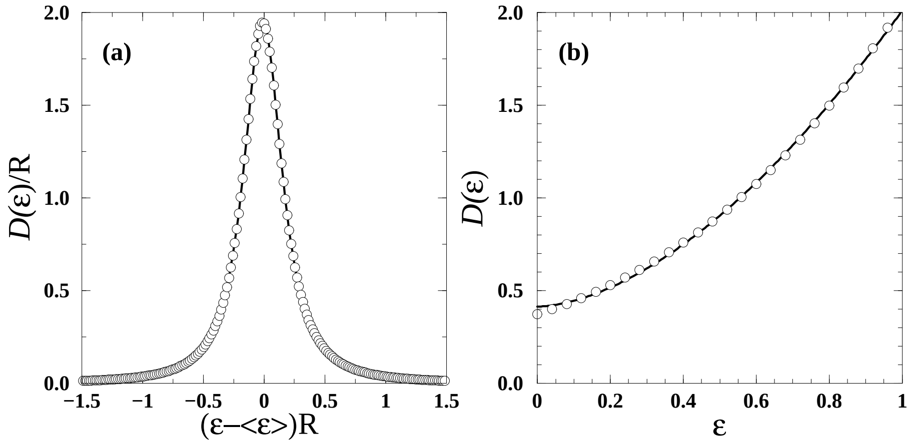

FIG. 2. $D(\epsilon)$ for (a) LAM and (b) BAM. The continuous curves are fits to a generalized Lorentzian function in (a) and to a power law in (b).

The size of the avalanche is generally measured by the total number of relaxations (s), the life-time of the avalanche (t) and the radius of the avalanche (r). Using $\{x : s, t, r\}$ we assume $x \sim x'^{\gamma_{x'x}}$, where the γ_{rx} is the cut-off exponent for the measure x, and the usual finite size scaling forms for the probability distribution functions of avalanche sizes: $\text{Prob}(x, L) \sim L^{-\beta_x} f_x\left(\frac{x}{L^{\gamma_{rx}}}\right)$, where, $f_x(y) \sim y^{-\tau_x}$ in the limit of $y \to 0$ gives $\tau_x = \beta_x/\gamma_{rx}$, in case Prob does not maintain an L dependence for $s << L$. The γ exponents are connected by the relations: $\gamma_{x'x} = \gamma_{x'x''}\gamma_{x''x}$ and $\gamma_{xx'} = (\tau_{x'} - 1)/(\tau_x - 1)$.

The steady state of the LAM is related to the first passage problem of RW's. The avalanche size and the life-time are the same as the number of steps s taken by the walker before dropping through the boundary. The distribution of the avalanche sizes can be calculated in the following way. The probability to start a RW within a distance ΔR from the boundary is $2\pi R\Delta R/\pi R^2 \sim \Delta R/R$. A RW moves a distance $\sim s^{1/2}$ in s steps. Therefore, the probability that an arbitrary walker makes s steps or less before reaching the boundary is the probability that an arbitrary site is within a distance $s^{1/2}$ from the boundary, which is $\sim s^{1/2}/R$. The probability that an arbitrary walker makes precisely s steps to reach the boundary is $s^{-1/2}/R$. Thus, in terms of L, $\text{Prob}(s, L) \sim s^{-\tau_s}/L$ for $s << L$ with $\tau_s = 1/2$. Our simulations show that upon increasing the system size the effective exponent $\tau_s(L)$ gradually decreases to its asymptotic limit of 0.50(1). Since the avalanches are random walks, the cut-off for s must be proportional to L^2, and therefore $\gamma_{rs} = 2$. Normalization of $\text{Prob}(s, L)$ gives $\beta_s = 2$. The relation $\tau_s = \beta_s/\gamma_{rs}$ is not satisfied by the LAM avalanches, because Prob maintains an L dependence for $s << L$. This unusual dependence is due to the fact that LAM avalanches must necessarily reach the boundary in order to extinguish. As a rule, this is not the case for other SOC systems.

We compare these results with two cases studied in the literature. A model of SOC where avalanches are branchless walks is the Eulerian walkers model. In this model each site of a regular lattice has an outgoing direction, which is one member of a set of outgoing bonds associated with this site. The walker leaves the site along the outgoing direction but changes the outgoing direction sequentially to the next member in the set of outgoing bonds [12]. In the Euler walker case, the dependence of $\text{Prob}(s, L)$ is of the form $L^{-2}f(sL^{-2})$ just similar to our LAM where s is the number of steps to the boundary. Secondly for the random walks on a square with absorbing boundary, the exact scaling function for the distribution of number of steps to the boundary is $P(s) \sim L^{-2}f(sL^{-2})$ has been calculated. It has been shown that the scaling function $f(x)$ can be expressed explicitly in terms of the Jacobi theta function [15].

Next we study a situation when the discharge ratio α is a random variable and a fresh value for it is drawn from a uniform random distribution in $\{0, 1\}$ each time a site relaxes. Since now the energy released after activation may be arbitrarily small, multiple relaxations are quite frequent, and this leads to branching. We call this the branched avalanche model (BAM). The probability density $D(\epsilon)$ of the site energies in the steady state has now little dependence on δ as well as on L. $D(\epsilon)$ is fitted best to a form $D(\epsilon) = D_o + D_1\epsilon^\mu$, with $D_o = 0.41(1)$, $D_1 = 1.59(1)$ and $\mu = 1.70(2)$ (Fig. 2). The average energy per site $\langle\epsilon\rangle$ is however observed to have an L dependence as: $\langle\epsilon\rangle = \epsilon_\infty - CL^{-3/4}$ with $\epsilon_\infty = 0.6365(3)$ and $C = 0.21(2)$.

The probability distributions for avalanche sizes and life-times follow the finite size scaling form of Prob very well. The data for $\text{Prob}(s, L)$ for three different system sizes $L = 129$, 513 and 2049 collapse well for $\gamma_{rs} = 2.75$ and $\beta_s = 3.6$. This gives a value for $\tau_s = 1.31$. Similarly, we obtain $\tau_t = 1.51(3)$ and $\gamma_{rt} = 1.50(3)$. The average

size $\langle s(L)\rangle \sim L^{\nu_s}$ with $\nu_s = 2$ and similarly $\nu_t = 0.76(3)$ are obtained. The values of these exponents are very close to those of the two-state model, indicating that the BAM may well belong to the universality class of the two-state model [5]. In the BAM, avalanches can extinguish within the boundary.

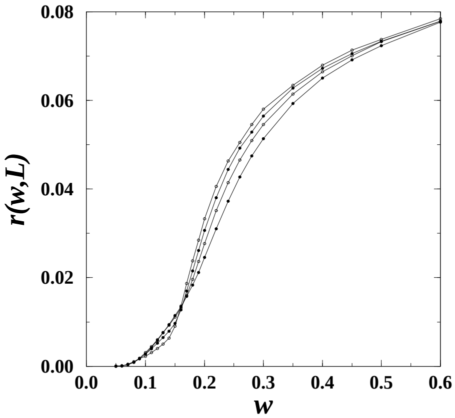

FIG. 3. Variation of the average branching number $r(w, L)$ with w. The curves become steeper as the system size L increases from 33 to 512. The threshold for BAM is at $w_c = 0.125 \pm 0.010$.

The LAM and the BAM can be regarded as the two extremes of a range of situations characterized by a progressively increasing width of the interval in which α is allowed to fluctuate. We introduce a parameter w as the size of the window for the stochastic α, which is now chosen randomly within the range $\{1/2 - w/2, 1/2 + w/2\}$ with uniform probability. When $w = 0$ we have the LAM with a constant $\alpha = 1/2$, whereas when $w = 1$ we have the BAM. A full characterization of all dynamical regimes for $0 < w < 1$ turns out to be extremely challenging. However, starting from $w = 1$ and progressively lowering this parameter, we are able to identify a whole region $w_c < w < 1$ in which BAM behaviour holds, and to characterize peculiar, novel scaling at the threshold for BAM behaviour, $w = w_c$. In the interval $0 < w < w_c$ for sure LAM behaviour prevails in a whole neighbourhood of $w = 0$. However, a sort of double degeneracy of the stationary state arises for higher w's, making the scaling analysis quite difficult. This degeneracy is an interesting phenomenon in itself, worth further investigations, and indicates that the LAM-BAM transition is a very complex process.

A lower bound for w_c can be estimated by calculating the maximum amount of energy a site can receive which is needed for at least two relaxations. Suppose all sites have the same energy $\langle\epsilon\rangle$ and $w_1 = 1/2 - w/2$ and $w_2 = 1/2 + w/2$. Then on adding an amount δ of energy at a site, the maximum amount of energy with which a site relaxes after the s-th step of a random walk is: $....(w_2(w_2(w_2(\delta + \langle\epsilon\rangle) + \langle\epsilon\rangle) + \langle\epsilon\rangle))..)... = \langle\epsilon\rangle(1 + w_2 + w_2^2 + w_3^3 + ... + w_2^s) + w_2^s\delta$ which in the $s \to \infty$ limit gives $\langle\epsilon\rangle/(1 - w_2)$. If this site now releases a fraction w_1 of its energy, the amount left is $\langle\epsilon\rangle((1 - w_1)/(1 - w_2))$ which has to be greater than one for a second relaxation to take place. Now since $\langle\epsilon\rangle$ can be at most $(1 - \delta)$, one must have $w_c > \delta/(2 - \delta)$.

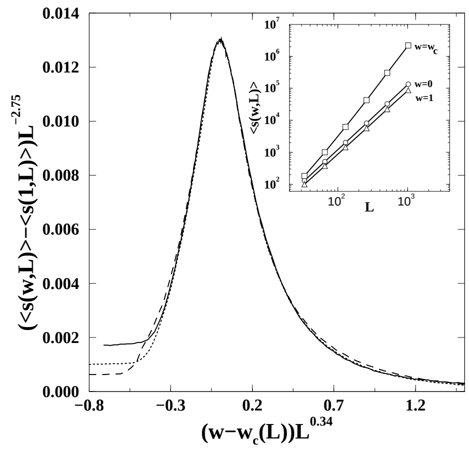

FIG. 4. The data collapse of the difference in the average size of the avalanche $\langle s(w, L)\rangle$ and that in the BAM $\langle s(1, L)\rangle$ is plotted for $L = 65, 129$ and 257. The inset shows the variation of $\langle s(w, L)\rangle \sim L^{\nu_s(w)}$ where $\nu_s(w) = 2.75$, 2.01, 2.00 for $w = w_c$, 0 and 1 respectively.

The basic difference in the limiting cases $w = 0$ and $w = 1$ is the branching of the avalanches. Therefore, a quantity which one can monitor in order to describe the passage from LAM to BAM behaviour is the ratio $\langle s\rangle/\langle t\rangle$. If on average an avalanche has size $\langle s\rangle$ and life-time $\langle t\rangle$, $\langle s\rangle/\langle t\rangle$ measures the number of branches, once assumed that each branch has a duration $\langle t\rangle$. Therefore, we define the scaled branch number as the order parameter:

$$r(w, L) = (\langle s\rangle/\langle t\rangle - 1)/L^{\gamma_{rs} - \gamma_{rt}} \tag{4}$$

and define it as an order parameter in the limit $L \to \infty$. By definition this order parameter must be finite and nonzero for $w > w_c$. It also turns out to be equal to zero for $w \le w_c$. Fig. 3 shows that for any L, $r(w, L)$ is very close to zero for $w < w_c$, but it increases very fast for $w > w_c$. Upon making L larger, $r(w, L)$ becomes progressively smaller for $w < w_c$, whereas it rises at a faster rate while $w > w_c$. Using $\delta = 1/8$ we identify $w_c = 0.125 \pm 0.005$. The sharp increase of $r(w, L)$ as $w \to w_c^+$ is well fitted as $r(w, L) \sim (w - w_c)^\beta$, with $\beta = 1.2 \pm 0.1$.

The average size of the avalanche $\langle s(w, L)\rangle$ has a sharp peak at $w_c(L)$ and it depends on L as: $w_c(L) = w_c + 0.39L^{-1/2}$ with $w_c = 0.123$. The data collapse well when $(\langle s(w, L)\rangle - \langle s(1, L)\rangle)L^{-2.75}$ is plotted versus $(w - w_c)L^{0.34}$. This indicates that $\langle s(w_c, L)\rangle \sim L^{2.75}$, which implies an anomalous diffusive transport at threshold (Fig. 4). Such anomalous transport has never been

reported before for a SOC model, to our knowledge. At $w = w_c$ the branch number $\langle s\rangle/\langle t\rangle$ grows as $L^{1.31}$. So, the threshold regime with anomalous diffusive transport is still characterized by infinite branching of the avalanches.

The dependence of $\langle s(w_c, L)\rangle \sim L^{2.75}$ at $w = w_c$ seems quite surprizing. We provide the following tentative explanation for the same: At any nonzero value of w some avalanches are purely random walks (i.e. $s = t$) and others are branched. If f_{RW} and $f_{BA} = 1-f_{RW}$ are the fractions of avalanches which are linear and branched, then for $w = 0$, f_{RW}=1 and it decreases as w increases, consequently f_{BA} increases from its zero value. Both fractions become very close to 1/2 at around $w = 0.4$ and beyond this value they vary slowly to $f_{RW} \approx 0.6$ and $f_{BA} \approx 0.4$ at $w = 1$. We also observed the fraction of RWs which terminate within the boundary of the system and donot drop out any energy outside the system. We find that the fraction of such random walk avalanches increases from zero very sharply to almost 0.9 at around w=0.15. This implies that around $w = w_c$ most of the δ input energies corresponding to those RWs which terminate within the boundary get stored in the system. Therefore, it is the branched avalanches which take out this extra stored energy from the system - consequently their sizes are bigger and perhaps for this reason the average size varies as a larger power of L like 2.75.

At $w = w_c$, the avalanche size distribution $D(s)$ vs. s has two regions with two characteristic sizes s_c^1 and s_c^2. While $s_c^2 \sim L^{2.75}$ is the usual cut-off size for the avalanches, $s_c^1 \sim L^2$ is an intermediate size. For $s < s_c^1$, $\tau_s^1 = 0.64$ whereas for the second region $s_c^1 < s < s_c^2$ the value of τ_s^2 is around 1.45. For $w < w_c$ but near to it, $s_c^1 \sim L^2$ whereas s_c^2 grows as w approaches w_c. However, for $w > w_c$, s_c^1 decreases as $s_c^1 \sim (w - w_c)^{-2.25}$ as the deviation $(w-w_c)$ increases and τ_s^2 also continuously decreases to 1.3, while $\tau_s^1 \to 0$ as $L \to \infty$.

To summarize, in a stochastic energy activation model an active site transfers the energy to only one randomly chosen nearest neighbour. The avalanches are linear when the transfer amount is narrow distributed and are branched when the transfer is broadly distributed. While the exponents of the toppling distribution of the branchless avalanches can be exactly determined, the universality class of the branched ones appears compatible with that of the stochastic two-state sandpile. A transition between the two regimes is observed by tuning the size of the window of the stochastic transfer ratio. At the transition point, the diffusion mechanism induced by avalanches is anomalous.

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